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LETTER TO THE EDITOR

Surface and edge exponents for the spreading of 3D percolation

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Abstract. Monte Carlo results are presented for the spreading of 3D percolation, where the seed consists of an (infinite) straight line. This line can either be in the bulk of the material into which spreading is possible, on a planar surface of the material, or on a rectangular edge. In the last two cases, spreading occurs only into angular regions with 180° (resp 90°). While the mean distance grows at $p = p_c$ in all three cases with the same exponent, the average number of growth sites grows (resp vanishes) in all three cases with different exponents. In particular, this implies that at $p = p_c$, the sites on the surface of a big cube belonging to the maximal cluster within that cube have fractal dimension <1.

One particular way of obtaining large percolation clusters consists in letting them grow or 'spread' from some seed, by 'infecting' in each time step all infectable neighbours, in the same way as an epidemic with a short infectious period spreads in a population, or a forest fire in a forest [1-3].

Apart from giving results on the time evolution, which is interesting in itself, this also seems to be one of the most efficient ways to study numerically geometrical aspects of ordinary (time-independent) percolation. Indeed, the critical exponents and critical values of p obtained in [4] for $d = 3$ and $d = 4$ with relatively modest effort are among the most precise ones obtainable today.

In [4], we studied bond percolation by starting with all sites on an entire hyperplane wetted (or 'infected') at time $t = 0$, such that the mean spreading is only in the direction perpendicular to that hyperplane. Numerically, this is advantageous over spreading from a single point: in the latter case, the average number of growth sites (i.e. sites infectious at the given time) grows very fast. Thus most of the time in a simulation is spent at late times, while fluctuations in the number of growth sites and in their average distance from the seed are dominated by what happened at earlier times. Also, when starting from a hypersurface, one can simulate much longer times on a lattice of given size, since the spreading is essentially in only one direction.

Unfortunately, however, starting from a hypersurface also has a drawback. In both three and four dimensions, the average number of growth sites now decreases strongly, such that most of the time is spent at early times, and fluctuations become large at late times. It would thus be advantageous if one had a way to grow the clusters such that the average number of growth sites stays roughly constant.

At least in three dimensions, this is the case for spreading from a straight line. Therefore, we shall present in this letter results from simulations of bond percolation on simple cubic lattices, with the seed being a straight line. In principle, this line

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should be infinitely long. We made sure that it was always much longer (by a factor of at least 2.3) than the distance of spreading perpendicular to the line.

In addition to placing the seed line into the middle of the lattice (called case I in the following), we also made runs where the line was on its boundary. When putting it in the middle of a surface, spreading occurred into half-space ('case II'). When putting it on an edge of the lattice, spreading was allowed only into an angular region with a 90° angle ('case III').

In all three cases, we can make a scaling ansatz for the density of growth sites formally identical to the ansatz made in [2]:

$$\rho(\boldsymbol{r}, t) = t^{-\delta} F(\varepsilon r^{1/\nu}, \varepsilon t^{1/\nu_t}) \qquad \text{with } \varepsilon = p - p_c. \tag{1}$$

The critical exponents ν and ν_t should be independent of the seed (they depend only on the dimension of the lattice). On the other hand, the scaling function F and the exponent δ will in general depend on the seed.

For a point seed, we must require that the density $\phi(\boldsymbol{r}, t)$ of wetted sites, given by

$$\phi(\boldsymbol{r}, t) = \int_0^t \mathrm{d}t' \, \rho(\boldsymbol{r}, t') \tag{2}$$

behaves for $p > p_c$ and for $r, t \to \infty$ like $\varepsilon^{2\beta}$. One factor ε^{β} comes from the probability that the seed is in the infinite cluster, the other from the probability that the point at $\boldsymbol{r}$ is. This gives then [2]

$$\delta_{\text{point seed}} = (2\beta/\nu_t) + 1. \tag{3}$$

For a hyperplane seed, the seed contains points in the infinite cluster with probability one, and an analogous consideration [2] gives

$$\delta_{\text{hyperplane}} = (\beta/\nu_t) + 1. \tag{4}$$

When applying these considerations to starting from a line seed, we have to know the probability that the seed contains points of the infinite cluster. In case I, there is no problem: since the infinite cluster has fractal codimension $d - d_F = 0.49$ [4], and thus a line will cut it with probability 1. We then obtain

$$\delta_{\text{case I}} = \delta_{\text{hyperplane}}. \tag{5}$$

In the other two cases, we cannot use this argument. Consider the maximal cluster in a large cube, after cutting off all tangles which are connected to it via paths outside the cube (see figure 1). The intersection between the surface of the cube and this truncated cluster is a set of fractal dimension d_F(surface). One might suspect that d_F(surface) $= d_F - 1$, since cutting a fractal by a hypersurface reduces its dimension by 1. But this is not true due to the omission of the tangles indicated in figure 1. If d_F(surface) > 1, a line will cut it with probability 1, and δ for case II is also given by (5). Otherwise, it will be larger for the same reason that $\delta_{\text{hyperplane}}$ is larger than $\delta_{\text{point seed}}$. The same considerations hold for case III, and we can only say that

$$\delta_{\text{case III}} \geqslant \delta_{\text{case II}} \geqslant \delta_{\text{case I}}. \tag{6}$$

Thus we must use our Monte Carlo simulations to decide this problem. For each seed and for each value of p, we made around 5000 independent runs. The number of time steps per run can be read off from the figures. The sizes of the lattices were $180 \times 170 \times 170$ (case 1), $240 \times 210 \times 106$ (case II) and $300 \times 136 \times 136$ (case III), with

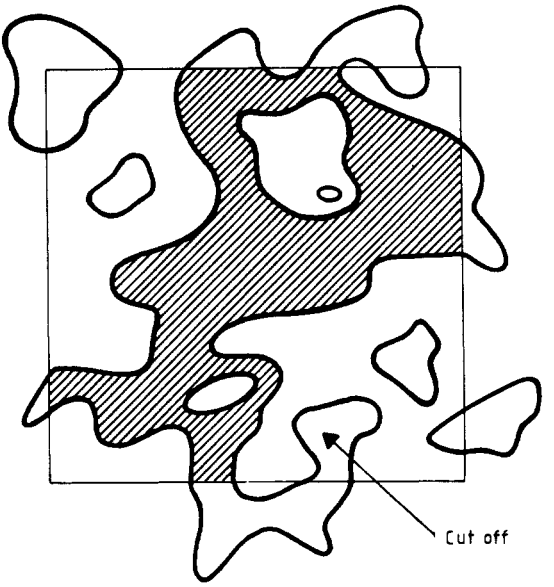

Figure 1. When considering only a finite part of a lattice (here two-dimensional), we cut off from the maximal cluster (shaded area) tangles which are connected to it only via bonds outside the considered part.

the seed always parallel to the x direction. Total CPU time spent was 30 h on a CYBER 170/175.

In figure 2, we show the average distances of growth sites from the seed as a function of time, for fixed $p \simeq p_c$. As in the case of spreading from a hypersurface [4], we found that these data do not depend strongly on p. In figures 3-5, we show

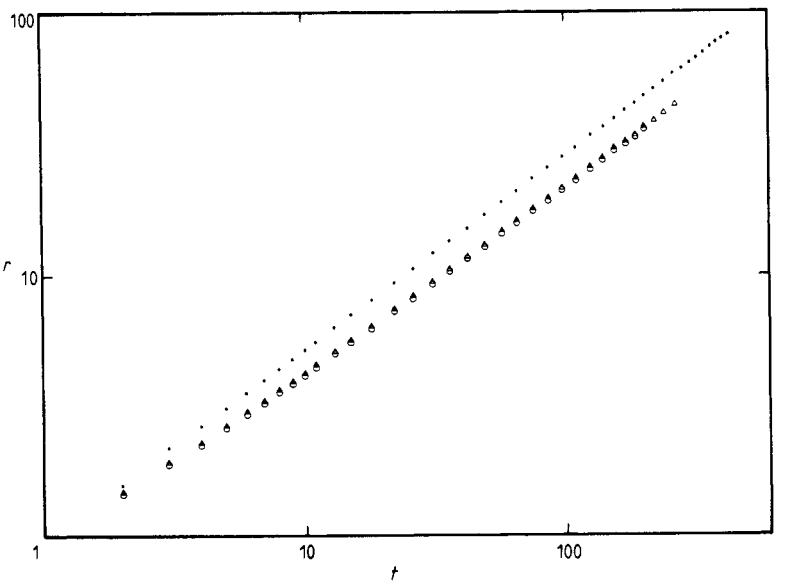

Figure 2. Average distance of growth sites for 3D bond percolation at $p = 0.2488$ plotted against time in a doubly logarithmic plot. The seed was a line through the middle of the lattice (○), on a surface (△), resp on an edge (●).

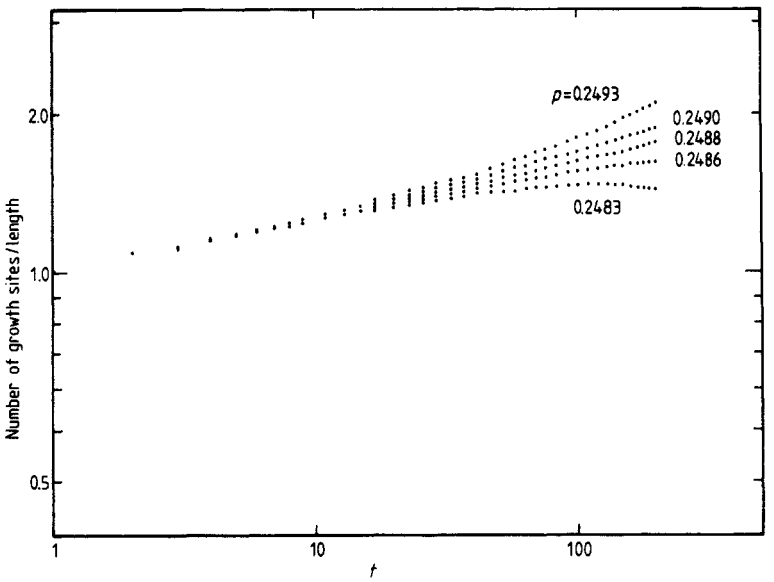

Figure 3. Average number of growth sites against time on a log-log plot, for line seeds passing through the centre of the lattice, and for various values of p.

the average numbers N of growth sites, divided by the length of the seeds. We find there a very strong dependence on p and also on the kind of seed. From the linearity of the curves in figures 3-5 and from more sophisticated analyses [4] we find the critical value of p to be at ~ 0.2488. Together with the somewhat lower but compatible data of [4], this gives

$$p_c = 0.248\,75 \pm 0.000\,13. \tag{7}$$

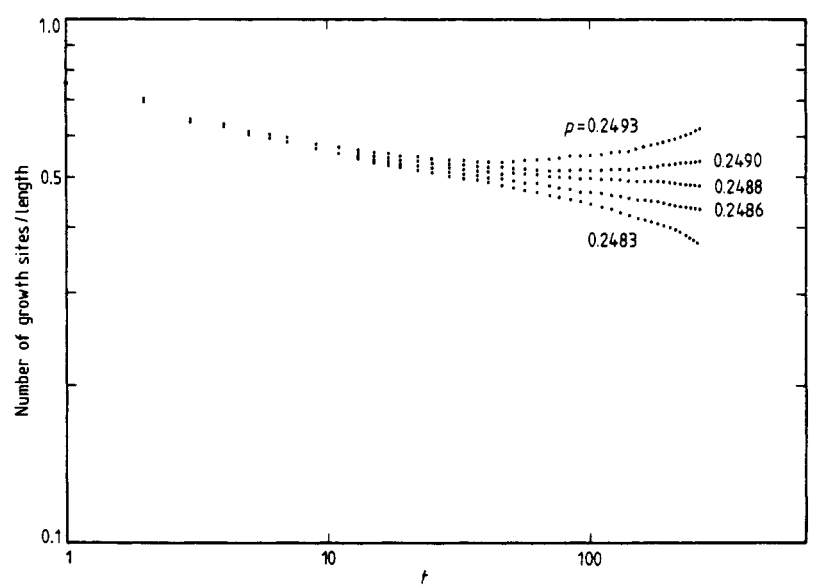

Figure 4. Same as figure 3, but for line seeds on a surface of the lattice.

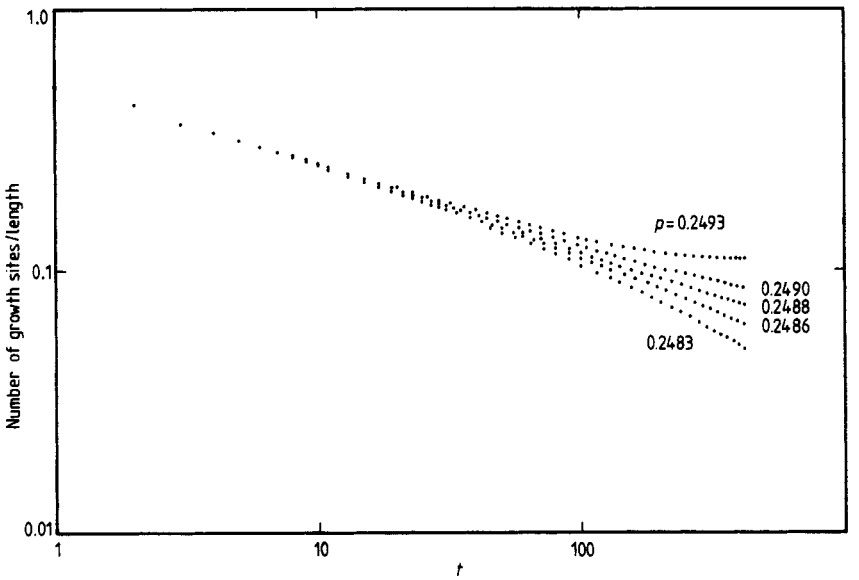

Figure 5. Same as figure 3, but for line seeds on an edge of the lattice.

From (1), we expect at $p = p_c$ the scaling laws [4]

$$r \sim t^{z_1} \qquad z_1 = \nu/\nu_t \tag{8}$$

$$N \sim t^{-z_2} \qquad z_2 = \delta - (2\nu/\nu_t). \tag{9}$$

As we see in the data, the critical exponent of r is indeed independent of the seed, and agrees well with that obtained in [4] (best overall estimate: $z_1 = 0.728 \pm 0.006$). For the exponent of N, we find in case I

$$z_2 = 1 + [(\beta - 2\nu)/\nu_t] = -0.11 \pm 0.01. \tag{10}$$

Again this is slightly but not seriously off the value obtained from [4]. For instance, we obtain $\eta = 2\beta/\nu - 1 = -0.04 \pm 0.04$ as our new best estimate, instead of -0.02 ± 0.04 in [4].

From figure 5, we see immediately that in case III we get a different exponent z_2, and thus a different δ. The situation is less clear in case II (figure 4), due to very strong corrections to scaling. Yet it seems that the same value of z_2 as in case I can be excluded. According to the above discussion, this shows that the largest clusters in the considered lattices cut the surface in sets of dimension less than 1.

More colourfully, one might say that it is impossible to hold a piece of nearly critical Swiss cheese by holding all points on its edge: with probability one, the edges does not contain any point of the infinite bulk piece, and the cheese would fall down (although one could hold it with a needle passing through it!).

Coming back from Swiss cheese to numbers, we find

$$\begin{aligned} z_2 &= 0.02^{+0.02}_{-?} \qquad \text{case II} \\ z_2 &= 0.30 \pm 0.02 \qquad \text{case III.} \end{aligned} \tag{11}$$

It would be interesting to have predictions for these exponents from field theory or from some other source, but at present we do not have any such independent estimates.

We also do not know how these exponents are related to other surface or edge exponents like β_{surface} (β_{edge}) which describe how the probability that a surface (edge) site belongs to the maximal cluster, for $p > p_c$.

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